EFFECT OF A WATER FILM ON THE STRUCTURE OF ICE GROWING BENEATH IT

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It is demonstrated that ice growing beneath a water film may take on either a transparent or an opaque form, depending on the thickness of the water film.

The growth and formation of ice structures in some cases occurs in the presence of a water film on the surface. For example, such growth occurs in icing of bodies in supercooled clouds [1, 2]. The structure of the ice growing beneath a water film depends on the character of gas-bubble formation and growth in the film. The solubility of air in ice is approximately three orders lower than its solubility in water [3]. Therefore, in crystallization of water, displacement of air takes place on the phase boundary; a supersaturation zone is created ahead of the moving crystallization front. When this supersaturation reaches a value of 10-40%, formation of air bubbles commences [3]. In the crystallization process these air bubbles are included within the ice and cause it to be opaque. Normally mechanical impurities contained in the water serve as gas-bubble-formation centers. The number of active bubble-formation centers increases sharply with increase in supersaturation. This supersaturation, in turn, is determined by the intensity of air liberation from the crystallizing solution and its discharge from the crystallization front. If flow of a water film then occurs, as is often the case in icing of bodies in supercooled clouds, the gas-diffusion process in the film depends strongly on the flow regime.

Thus, the concentration and dimensions of the air bubbles formed in the water ahead of the moving phase boundary and, consequently, the opacity of the ice formed depend on the displacement rate of this phase boundary and on the thickness and flow regime of the water film [1].

To analyze the character of the gas distribution in the liquid film, we assume that ice growth occurs in the positive direction along the x axis at a constant velocity u, while the film thickness h remains constant.

Such an ice growth regime may occur, for example, in icing of bodies in a flow of supercooled aerosol. In particular, under natural conditions such a regime is realized in the wet growth of hailstones in a cloud [4]. If the body is located in a flow of supercooled aerosol, the mass flow density on its surface will be

$$q = \frac{4}{3}\pi\rho V n \int_0^\infty r^3 E(r) f(r)\, dr. \tag{1}$$

As is well known [5], the wet ice growth regime in a supercooled aerosol occurs when q is higher than some critical value q_c, determined by the condition that during the ice growth process the temperature of its surface remains equal to zero constantly. In this case the thermal balance equation with consideration of water evaporation from the ice surface takes on the form

$$Lq_c = \lambda_1 \frac{dT_1}{dx} + \alpha(T_0 - T_\infty) + c_w q_\infty (T_0 - T_\infty) + \beta L_1 (\bar{\rho}_s - \rho_\infty). \tag{2}$$

If the surrounding air is close to saturation, then in the first approximation we may write [4]

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$$\rho_s - \rho_\infty = \delta(T_s - T_\infty), \tag{3}$$

where $\delta \simeq 2.3 \cdot 10^{-7}$ g/cm³·deg. It follows from Eqs. (1) and (2) that

$$q_c = \frac{\lambda_1 \frac{dT_1}{dx} + (\alpha + \beta L_1 \delta)(T_0 - T_\infty)}{L - c_w(T_0 - T_\infty)}.$$

For $q > q_c$ a liquid film appears on the surface of the growing ice. For every value q of each combination of parameters α, β, T_∞, dT_1/dx there correspond definite values of u and h. In order to establish the dependence of u and h on q for the stationary state, we write the heat and mass balance equation for the water film.

On the film surface we have

$$-\lambda \frac{dT}{dx} = \alpha(T_s - T_\infty) + c_w q (T_s - T_\infty) + \beta L_1 (\bar{\rho}_s - \rho_\infty), \tag{4}$$

$$\rho u = q - \beta(\bar{\rho}_s - \rho_\infty). \tag{5}$$

On the water–ice phase boundary

$$T = T_1 = T_0, \tag{6}$$

$$\rho L u = -\lambda \frac{dT}{dx} + \lambda_1 \frac{dT_1}{dx}. \tag{7}$$

If we take a steady-state temperature distribution law in the liquid film, then

$$\frac{dT}{dx} = \frac{T_s - T_0}{h}. \tag{8}$$

It follows from Eqs. (3)-(7) that

$$T_s = T_\infty + \frac{Lq - \lambda_1 \left(\frac{dT_1}{dx}\right)}{\alpha + c_w q + \beta L_1 \delta + \beta L \delta}, \tag{9}$$

$$h = \lambda \frac{(\alpha + c_w q + \beta L_1 \delta + \beta L \delta)(T_0 - T_\infty) + \lambda_1 \frac{dT_1}{dx} - Lq}{(\alpha + c_w q + \beta L_1 \delta)\left(Lq - \lambda_1 \frac{dT_1}{dx}\right)}, \tag{10}$$

$$u = \frac{q}{\rho} - \frac{\beta \delta}{\rho} \cdot \frac{Lq - \lambda_1 \frac{dT_1}{dx}}{\alpha + c_w q + \beta \delta (L + L_1)}. \tag{11}$$

In deriving Eqs. (9)-(11) it was assumed that the liquid temperature on the crystallization front is equal to the stable ice–water phase equilibrium temperature.

In [1] the established liquid film thickness was calculated for the case of a turbulent flow regime, where the crystallization front velocity is a specified function of the temperature of the supercooled water in the film.

Thus, by varying the values of the parameters α, β, q, and dT_1/dx a steady-state ice growth regime may be established below the constant-thickness liquid film.

We will assume that in the crystallization process the liquid releases all the gas contained within it. In view of the small size of h, the effect of water-film surface curvature on gas concentration within the film may be neglected. Then in the coordinate system wherein the phase boundary x = 0 remains fixed, the equation for gas concentration c in the liquid film in the steady state may be written in the form

$$D \frac{d^2 c}{dx^2} + u \frac{dc}{dx} = -uc\delta(x). \tag{12}$$

The boundary condition on the film surface has the form

$$-D\left(\frac{dc}{dx}\right)_{x=h} = \beta(\rho_s - \rho_\infty). \tag{13}$$

The partial pressure P of the gas in equilibrium with its solution in the liquid, according to Henry's law, is given by the formula [6]

$$P = Kc_0. \tag{14}$$

Using Eq. (14), Eq. (13) may be written in the form

$$-D\left(\frac{dc}{dx}\right)_{x=h} = \frac{\beta}{\alpha'}(c_s - c_0), \tag{15}$$

where $\alpha' = K\mu/RT_\infty$ is the gas volume (in liters) dissolved in 1 liter of water.

Integrating Eq. (12) from $-\varepsilon$ to $+\varepsilon$, we obtain

$$uc\Big|_{-\varepsilon}^{+\varepsilon} + D\frac{dc}{dx}\Big|_{-\varepsilon}^{+\varepsilon} = -uc|_{x=0}.$$

Hence in the limit as $\varepsilon \to 0$ we will have

$$-D\left(\frac{dc}{dx}\right)_{x=0} = 2uc|_{x=0}. \tag{16}$$

For $x > 0$, Eq. (12) has the form

$$D\frac{d^2c}{dx^2} + u\frac{dc}{dx} = 0. \tag{17}$$

The solution of Eq. (17) satisfying boundary conditions (15) and (16) has the form

$$c = c_0\frac{2\exp\left(-\frac{ux}{D}\right) - 1}{2\left(1 - \frac{u\alpha'}{\beta}\right)\exp\left(-\frac{uh}{D}\right) - 1}. \tag{18}$$

It follows from Eq. (18) that a steady-state gas concentration distribution in the liquid film is possible only if the condition

$$h < \frac{D}{u}\ln\left[2\left(1 - \frac{u\alpha'}{\beta}\right)\right] \tag{19}$$

is fulfilled. We denote by c_c the critical gas concentration at which bubble formation commences in the water; the corresponding gas solution supersaturation will be $f_c = c_c/c_0$.

As follows from Eq. (18), the critical film thickness h_c, at which the phase boundary concentration $c = c_c$, is determined by the formula

$$h_c = \frac{D}{u}\ln\left[\frac{2f_c\left(1 - \frac{u\alpha'}{\beta}\right)}{1 + f_c}\right]. \tag{20}$$

For $h < h_c$ air bubbles will not be formed in the liquid film. It follows from Eq. (9) that the basic factor determining the magnitude of h_c is the ratio D/u.

If Eq. (19) is not fulfilled, then the concentration distribution in the liquid film will be non-steady-state; this concentration will increase in the ice growth process.

In the general case the non-steady-state concentration field in the liquid film will be described by the equation

$$\frac{\partial c}{\partial t} - D\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} = uc\delta(x). \tag{21}$$

Introducing the new function $c_1 = c - c_0$, we rewrite Eq. (21) in the form

$$\frac{\partial c_1}{\partial t} - D\frac{\partial^2 c_1}{\partial x^2} - u\frac{\partial c_1}{\partial x} = u(c_1 + c_0)\delta(x). \tag{22}$$

At the initial moment let the concentration $c = c_0$. Then at $t = 0$, $c_1 = 0$, and the boundary condition at $x = h$ takes on the form

$$-D\left(\frac{\partial c_1}{\partial x}\right)_{x=h} = \frac{\beta}{\alpha'}c_1|_{x=h}. \tag{23}$$

Taking the Laplace transform of Eq. (22) for the transform $\bar{c}$, we obtain the following equation:

$$\frac{d^2\bar{c}_1}{dx^2}+\frac{u}{D}\cdot\frac{d\bar{c}_1}{dx}-\frac{p}{D}\bar{c}_1=-\frac{u}{D}\left(\bar{c}_1+\frac{c_0}{p}\right)\delta(x). \tag{24}$$

As above, from Eq. (24) we have

$$-D\left(\frac{d\bar{c}_1}{dx}\right)_{x=0}=2u\left(\bar{c}_1+\frac{c_0}{p}\right)\Big|_{x=0}; \tag{25}$$

for x > 0, Eq. (24) has the form

$$\frac{d^2\bar{c}_1}{dx^2}+\frac{u}{D}\cdot\frac{d\bar{c}_1}{dx}-\frac{p}{D}\bar{c}_1=0. \tag{26}$$

The general solution of Eq. (26) satisfying boundary conditions (23) and (25) can be written in the form

$$\bar{c}=\frac{c_0}{\frac{\beta}{u\alpha'}-2}\exp\left(-\frac{ux}{D}\right)\left\{\frac{\left(\frac{2\beta}{\alpha'}-u\right)\exp\left(-\frac{\sqrt{p+\frac{u^2}{4D}}}{\sqrt{D}}x\right)}{2Dp\sqrt{p+\frac{u^2}{4D}}}+\exp\left(-\frac{\sqrt{p+\frac{u^2}{4D}}}{\sqrt{D}}x\right)+\right.$$

$$\left.\frac{1}{p}\exp\left[\frac{\sqrt{p+\frac{u^2}{4D}}}{\sqrt{D}}(2h-x)\right]-\frac{\frac{2\beta}{\alpha'}-u}{2\sqrt{D}}\cdot\frac{\exp\left[-\frac{\sqrt{p+\frac{u^2}{4D}}}{\sqrt{D}}(2h-x)\right]}{p\sqrt{p+\frac{u^2}{4D}}}\right\}. \tag{27}$$

Using the displacement theorem and multiplication [7], from solution (27) we easily find the original function $c_1(x, t)$ [8]. The final solution for c(x, t) has the form

$$c=c_0+\frac{uc_0}{2\sqrt{\pi D}}e^{-\frac{ux}{2D}}\left[\int_0^t\frac{e^{-\frac{x^2}{4D\tau}}+e^{-\frac{(2h-x)^2}{4D\tau}}}{\sqrt{\tau}}e^{-\frac{u^2\tau}{4D}}d\tau+\frac{h}{\frac{\beta}{2\alpha'}-u}\int_0^t\frac{e^{-\frac{x^2}{4D\tau}}+e^{-\frac{(2h-x)^2}{4D\tau}}}{\tau\sqrt{\tau}}e^{-\frac{u^2\tau}{4D}}d\tau\right]. \tag{28}$$

In the presence of wave motion in the liquid film, according to Kapitsa's theory, the coefficient D increases by approximately 20% in comparison to purely laminar flow. In the case of turbulent flow, instead of D we must take an effective coefficient of turbulent diffusion D_T.

Thus, in ice growth beneath a water film the formation of two types of ice structure is possible: transparent or opaque. If for a given ice growth rate the film thickness $h < h_c$, transparent ice is produced. For $h > h_c$, opaque ice is formed containing many bubbles. The concentration and dimensions of these bubbles depend on the ice growth rate and the thickness and flow regime of the water film. With change in thickness h a transition occurs from one form of crystal structure to the other. Therefore, the structure of the ice growing beneath the film may be controlled by changing the thickness of the liquid film.

NOTATION

x, coordinate; t, time; h, liquid film thickness; u, crystal growth rate; q, mass flow density; ρ, liquid density; V, gas flow velocity; n, r, concentration and radius of aerosol particles; E(r), aerosol particle coagulation coefficient with body surface; f(r), aerosol particle distribution function over size; L, L_1, specific heats of crystallization and evaporation of liquid; λ, λ_1, thermal-conductivity coefficients of water and ice; α, β, heat- and mass-transfer coefficients; T, liquid temperature in film; T_1, ice temperature; T_0, water–ice stable equilibrium temperature; T_s, T_∞, temperatures of film surface and surrounding medium; c_w, heat capacity of water; ρ_s, ρ_∞, vapor densities; ρ_s, ρ_∞, surrounding gas densities at the liquid film surface and far from it; c, c_s, gas concentrations in liquid film and on its surface; c_0, saturated gas density in liquid; D, gas diffusion coefficient in liquid; K, Henry constant; R, universal gas constant; μ, molecular weight of gas.

LITERATURE CITED

1. L. G. Kachurin and V. G. Morachevskii, Kinetics of Water Phase Transitions in the Atmosphere [in Russian], Izd. Leningr. Gos. Univ., Leningrad (1965).
2. I. P. Mazin, Physical Bases of Airplane Icing [in Russian], Gidrometeoizdat, Moscow (1957).
3. A. E. Carte, "Air bubbles in ice," Proc. Phys. Soc., 77, No. 495, 757-768 (1961).
4. M. K. Zhekamukhov, "A theory of hailstone growth in clouds," Tr. VGI, No. 19 (1971).
5. B. J. Mason, Cloud Physics [Russian translation], Gidrometeoizdat, Leningrad (1961).
6. E. A. Moelwyn-Hughes, Physical Chemistry, 2nd ed., Pergamon (1964).
7. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of a Complex Variable [in Russian], GIFML, Moscow (1958).
8. H. Bateman and A. Erdelyi (editors), Tables of Integral Transforms, McGraw-Hill (1954).

NUMERICAL SOLUTION OF LIPPMANN–SCHWINGER INTEGRAL EQUATION

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A method is discussed for obtaining a numerical solution of an equation similar to the equation for the transport of radiant energy for a steady radiation field [1].

The integral equation for thermal radiation, taking account of scattering, has the form [2]

$$t_l(p; p_1; k^2) = V_l(p; p_1) + \frac{2}{\pi}\int_0^\infty \frac{V_l(p; p_2)\, t_l(p_2; p_1; k^2)\, p_2^2}{k^2 - p_2^2 + i\varepsilon}\, dp_2, \tag{1}$$

where $t_l(p; p_1; k^2)$ is the partial probability amplitude for the scattering of a wave packet with energy k^2; p and p_1 are the magnitudes of the momenta of the wave packet before and after scattering; and ε is an infinitesimal indicating the rule for going around the contour of integration in the complex p_2 plane.

The kernel of Eq. (1) contains the function

$$V_l(p; p_1) = \int_0^\infty j_l(pr)\, V(r)\, j_l(p_1 r)\, r^2 dr, \tag{2}$$

where $j_l(pr)$ is a spherical Bessel function, and V(r) is a function characterizing the steady perturbing field. We consider a V(r) of the form

$$V(r) = V_1(r) + V_2(r), \tag{3}$$

where $V_1(r)$ is the positive function

$$V_1(r) = \begin{cases} V_0^1 \gg k^2, & 0 < r < r_c, \\ 0, & 0 > r_c, \end{cases} \tag{4}$$

and $V_2(r)$ is a negative function of two types:

$$V_2(r) = \begin{cases} -V_0, & r_c < r < r_0, \\ 0, & r > r_0 \end{cases} \tag{5}$$

or

$$V_2(r) = -V_0[1 + \exp((r - r_0)/a)]^{-1}, \quad r_c < r < \infty. \tag{6}$$

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